1 For posetal categories with initial objects, the pushout-conservative cocompletion distributes over the product

Thank you to Sam Staton and Owen Lynch for showing me a sketch proof of this result!

1.1 Preliminaries

Denote by $\mathbb{2}$ the preorder $\{0 \leq 1\}$ of truth values, viewed as a category.

Note that if a category is a poset, then it is in particular a preorder; preorders are $\mathbb{2}$-enriched categories [2].

For the rest of this section, let $\mathcal{A}$ be an arbitrary finite posetal category with an initial object a_0, which we may view as a $\mathbb{2}$-enriched category.

There is an isomorphism between downsets of $\mathcal{A}$ and objects in the functor category $[\mathcal{A}^{\text{op}}, \mathbb{2}]$. In particular, for any downset, define a functor by sending elements of the downset to 1 and the rest to 0; for any functor, define a downset by including all the objects sent to 1. Also note that arrows in this functor category are set inclusions between downsets. So the functor category is isomorphic to the posetal category of the set of all downsets of $\mathcal{A}$.

Note that $\mathbb{2}$ is a complete chain, and therefore is a completely distributive lattice; equalities using this fact will be marked $=^{\dagger}$.

1.2 $\cdot \leftarrow \cdot \rightarrow \cdot$-colimits are joins and $\cdot \rightarrow \cdot \leftarrow \cdot$-limits are meets in $\mathcal{A}$

Since $\mathcal{A}$ is thin, any triangle of arrows commutes.

So the colimit of a diagram $y \leftarrow x \rightarrow z$ does not depend on x; in fact, it is the join of y and z in the poset $\mathcal{A}$.

Similarly, the limit of a digram $b \rightarrow a \leftarrow c$ does not depend on a; it is the meet of b and c in $\mathcal{A}$.

1.3 Description of the pushout-conservative cocompletion of $\mathcal{A}$

Kelly [1] gives a description for the R-conservative cocompletion of an arbitrary category, where R is some set of diagram schemes. We aim to make this concrete in the case of $\mathcal{A}$ by using the description in [3]. Fix $R = \{\cdot \leftarrow \cdot \rightarrow \cdot\}$ from this point onwards. We would like to find a concrete description for $\text{CPsh}(\mathcal{A})$, the pushout-conservative cocompletion of $\mathcal{A}$. Note that since $\mathcal{A}$ has an initial object, the condition of $\text{CPsh}(\mathcal{A})$ having all colimits is equivalent to the condition of it having all pushouts. So we may also view $\text{CPsh}(\mathcal{A})$ as the pushout-preserving pushout completion.

1.3.1 Description of $[\mathcal{A}^{\text{op}}, \mathbb{2}]_{R^{\text{op}}}$

$[\mathcal{A}^{\text{op}}, \mathbb{2}]_{R^{\text{op}}}$ is the category of all functors $\mathcal{A}^{\text{op}} \rightarrow \mathbb{2}$ which preserve $\cdot \rightarrow \cdot \leftarrow \cdot$-limits. That is, these are functors sending $\cdot \leftarrow \cdot \rightarrow \cdot$-colimits in $\mathcal{A}$ to $\cdot \rightarrow \cdot \leftarrow \cdot$-limits in $\mathbb{2}$. But since both $\mathcal{A}$ and $\mathbb{2}$ are posets and by the description of limits/colimits of these diagram schemes, this is equivalent to functors which send joins in $\mathcal{A}$ to meets in $\mathbb{2}$. So the functors in $[\mathcal{A}^{\text{op}}, \mathbb{2}]_{R^{\text{op}}}$ are exactly those downsets that have an associated function $F : \text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathbb{2})$ on objects satisfying

$$\text{if } a \vee b \text{ is defined, then } F(a \vee b) = F(a) \wedge F(b) \tag{1}$$

(note that the only case in which this is a stronger condition than the condition of being a downset is when $F(a) = F(b) = 1$, in other words we additionally require that when both a and b are in the downset, so is their join).

1.3.2 Downsets generated by objects of $\mathcal{A}$

A functor in $[\mathcal{A}^{\text{op}}, \mathbb{2}]$ generated by an object z of $\mathcal{A}$ is the downset of all elements x such that $x \leq z$; we denote such a functor/downset by $\downarrow z$. Any such functor is also in $[\mathcal{A}^{\text{op}}, \mathbb{2}]_{R^{\text{op}}}$ since if any two elements a and b are in the downset, then they are comparable (by the existence of the bottom/initial element), and so their join is their maximum.

1.3.3 Closure under all small colimits

Finally, to obtain the pushout-conservative cocompletion of $\mathcal{A}$, we close $\mathcal{A}$ in $[\mathcal{A}^{\text{op}}, \mathbb{2}]_{R^{\text{op}}}$ under all (small) colimits. We aim to show that this closure consists of the subcategory of exactly those functors F that correspond to non-empty downsets satisfying (1).

The subcategory is closed under all non-empty colimits (arguing by contradiction, there is no inclusion arrow from any non-empty downset to the empty downset), and the colimit of the empty diagram is $\downarrow a_0$, the initial object in the subcategory we have defined (recall that a_0 is the initial object of $\mathcal{A}$).

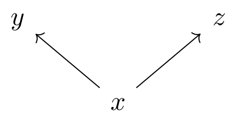
The subcategory is also generated by representables, since any desired downset $F = \{x_1, x_2, ..., x_k\}$ is the colimit of the diagram $(\downarrow x_1 \quad \downarrow x_2 \quad ... \quad \downarrow x_k)$.

1.3.4 Description of CPsh $(\mathcal{A})$

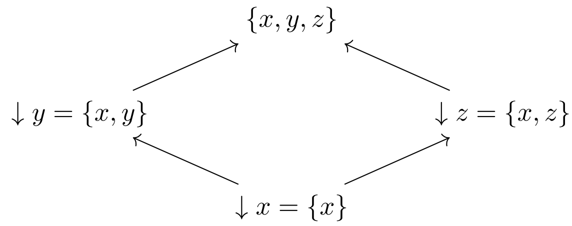
We conclude that CPsh $(\mathcal{A})$ consists of the poset of all non-empty downsets of $\mathcal{A}$ which satisfy (1), ordered by the set inclusion relation.

1.3.5 Examples

i) The pushout-conservative cocompletion of

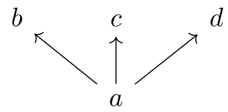


is

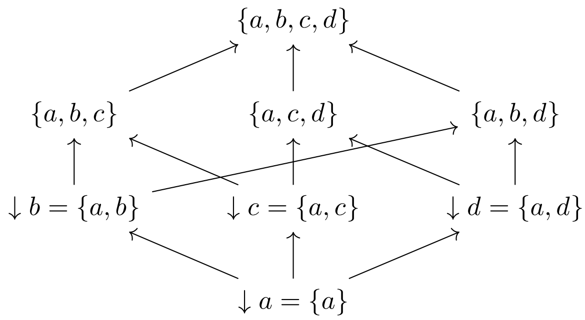


since the condition (1) is satisfied trivially by any downset.

ii) The pushout-conservative cocompletion of

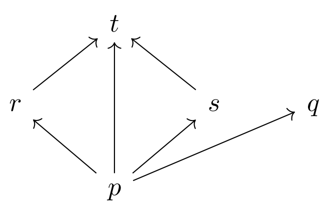


is the transitive closure of

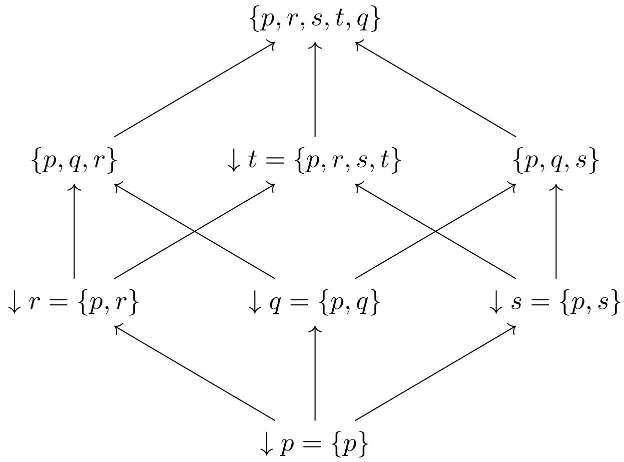


since the condition (1) is satisfied trivially by any downset.

iii) The pushout-conservative cocompletion of



is the transitive closure of



where the condition (1) forbids the inclusion of $\{p, r, s\}$ and $\{p, r, s, q\}$.

1.4 Statement of the claim

Let $\mathcal{C}$ and $\mathcal{D}$ be finite posetal categories, each with an initial element. Then, their pushout-conservative cocompletions distribute over taking the product.

That is, $\mathrm{CPsh}(\mathcal{C} \times \mathcal{D}) \cong \mathrm{CPsh}(\mathcal{C}) \times \mathrm{CPsh}(\mathcal{D})$.

1.5 Proof of the claim

Define the functors $G : \mathrm{CPsh}(\mathcal{C}) \times \mathrm{CPsh}(\mathcal{D}) \to \mathrm{CPsh}(\mathcal{C} \times \mathcal{D})$ and $H : \mathrm{CPsh}(\mathcal{C} \times \mathcal{D}) \to \mathrm{CPsh}(\mathcal{C}) \times \mathrm{CPsh}(\mathcal{D})$ by

$$\begin{aligned} G(F_1, F_2) &= (p_1, p_2) \mapsto F_1(p_1) \wedge F_2(p_2) \,, \\ H(F) &= \left(p_1 \mapsto \bigvee_{p_2 \in \mathrm{ob}(\mathcal{C})} F(p_1, p_2), p_2 \mapsto \bigvee_{p_1 \in \mathrm{ob}(\mathcal{D})} F(p_1, p_2) \right) \,. \end{aligned}$$

1.5.1 For $F_1 \in \mathbf{CPsh}(\mathcal{C})$ and $F_2 \in \mathbf{CPsh}(\mathcal{D})$, we have $G(F_1, F_2) \in \mathbf{CPsh}(\mathcal{C} \times \mathcal{D})$

i) Suppose we have $(q_1, q_2) \leq (p_1, p_2)$ in $\mathcal{C} \times \mathcal{D}$, and $G(F_1, F_2)(p_1, p_2) = 1$.
Then $q_1 \leq p_1$ and $q_2 \leq p_2$ and $F_1(p_1) = F_2(p_2) = 1$.
So $F_1(q_1) = F_2(q_2) = 1$ and $G(F_1, F_2)(q_1, q_2) = 1$.
So $G(F_1, F_2)$ is a downset, and it's clearly non-empty by the non-emptiness of each of F_1 and F_2.

ii) If F_1 and F_2 satisfy (1), then

$$\begin{aligned} G(F_1, F_2)((p_1, p_2) \vee (q_1, q_2)) &= G(F_1, F_2)(p_1 \vee q_1, p_2 \vee q_2) \\ &= F_1(p_1 \vee q_1) \wedge F_2(p_2 \vee q_2) \\ &= F_1(p_1) \wedge F_1(q_1) \wedge F_2(p_2) \wedge F_2(q_2) \\ &= G(F_1, F_2)(p_1, p_2) \wedge G(F_1, F_2)(q_1, q_2) \end{aligned}$$

and we conclude that $G(F_1, F_2)$ satisfies (1).

1.5.2 For $F \in \mathbf{CPsh}(\mathcal{C} \times \mathcal{D})$, we have $H(F) \in \mathbf{CPsh}(\mathcal{C}) \times \mathbf{CPsh}(\mathcal{D})$

By symmetry it suffices to check that $\left(p_1 \mapsto \bigvee_{p_2 \in \mathrm{ob}(\mathcal{C})} F(p_1, p_2)\right) \in \mathrm{CPsh}(\mathcal{C})$.

i) Suppose we have $q_1 \leq p_1$ in $\mathcal{C}$, and $\bigvee_{p_2 \in \mathrm{ob}(\mathcal{C})} F(p_1, p_2) = 1$.
So $\exists p_2 \in \mathrm{ob}(\mathcal{C})$ such that $F(p_1, p_2) = 1$.
Since $(q_1, p_2) \leq (p_1, p_2)$ in $\mathcal{C} \times \mathcal{D}$, we have $F(q_1, p_2) = 1$ and so $\bigvee_{p_2 \in \mathrm{ob}(\mathcal{C})} F(q_1, p_2) = 1$.
Since F is non-empty, there is some (p_1', p_2') such that $F(p_1', p_2') = 1$. Then, taking p_1' as input suffices to check the non-emptiness of $\left(p_1 \mapsto \bigvee_{p_2 \in \mathrm{ob}(\mathcal{C})} F(p_1, p_2)\right)$.

ii) If F satisfies (1), then

$$\begin{aligned}\bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F(p_1\vee q_1,p_2) &= \bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F((p_1,p_2)\vee(q_1,p_2))\\ &= \bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F(p_1,p_2)\wedge F(q_1,p_2)\\ &=^{\dagger} \bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F(p_1,p_2)\wedge \bigvee_{p_2'\in\mathrm{ob}(\mathcal{C})} F(q_1,p_2')\end{aligned}$$

and we conclude that $H(F)$ satisfies (1).

1.5.3 $G\circ H$ is identity on objects of $\mathbf{CPsh}\,(\mathcal{C}\times\mathcal{D})$

We have shown above that $\mathrm{CPsh}\,(\mathcal{C}\times\mathcal{D})$ is posetal, so it suffices to show an inequality in each direction.

We have

$$\begin{aligned}(G\circ H)(F)(p_1,p_2) &= \bigvee_{p_2'\in\mathrm{ob}(\mathcal{C})} F(p_1,p_2')\wedge \bigvee_{p_1'\in\mathrm{ob}(\mathcal{D})} F(p_1',p_2)\\ &=^{\dagger} \bigvee_{p_2'\in\mathrm{ob}(\mathcal{C})}\bigvee_{p_1'\in\mathrm{ob}(\mathcal{D})} F(p_1,p_2')\wedge F(p_1',p_2)\\ &= \max_{p_2'\in\mathrm{ob}(\mathcal{C}),\,p_1'\in\mathrm{ob}(\mathcal{D})} F(p_1,p_2')\wedge F(p_1',p_2) &&\text{since } 2 \text{ is a total order}\\ &\geq F(p_1,p_2)\wedge F(p_1,p_2)\\ &= F(p_1,p_2)\end{aligned}$$

and further, for an arbitrary $p_2'\in\mathrm{ob}(\mathcal{C})$ and $p_1'\in\mathrm{ob}(\mathcal{D})$ we have $F(p_1,p_2')\leq F(p_1,\bot)$ and $F(p_1',p_2)\leq F(\bot,p_2)$ since F is a downset. So

$$\begin{aligned}F(p_1,p_2')\wedge F(p_1',p_2) &\leq F(p_1,\bot)\wedge F(\bot,p_2)\\ &\leq F((p_1,\bot)\vee(\bot,p_2))\\ &\leq F(p_1,p_2)\end{aligned}$$

and we conclude that $(G\circ H)(F)(p_1,p_2)=\max_{p_2'\in\mathrm{ob}(\mathcal{C}),\,p_1'\in\mathrm{ob}(\mathcal{D})}\left(F(p_1,p_2')\wedge F(p_1',p_2)\right)\leq F(p_1,p_2)$.

1.5.4 $H\circ G$ is identity on objects of $\mathbf{CPsh}\,(\mathcal{C})\times\mathbf{CPsh}\,(\mathcal{D})$

By symmetry it suffices to check the first component of $(H\circ G)(F_1,F_2)$, call it $((H\circ G)(F_1,F_2))_1$. We have

$$\begin{aligned}((H\circ G)(F_1,F_2))_1(p_1) &= \bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F_1(p_1)\wedge F_2(p_2)\\ &=^{\dagger} F_1(p_1)\wedge \bigvee_{p_2\in\mathrm{ob}(\mathcal{C})} F_2(p_2)\\ &= F_1(p_1)\wedge 1 &&\text{since } F_2 \text{ is a non-empty downset}\\ &= F_1(p_1)\end{aligned}$$

as required.

1.5.5 $G\circ H$ and $H\circ G$ are identities on arrows

Since $\mathrm{CPsh}\,(\mathcal{C}\times\mathcal{D})$ and $\mathrm{CPsh}\,(\mathcal{C})\times\mathrm{CPsh}\,(\mathcal{D})$ are thin categories, this condition is trivially satisfied.

References

[1] Gregory Maxwell Kelly. *Basic concepts of enriched category theory.* Vol. 64. CUP Archive, 1982. ISBN: 0-521-28702-2.

[2] *relation between preorders and (0,1)-categories in nLab.* URL: `https://ncatlab.org/nlab/show/relation+between+preorders+and+%280%2C1%29-categories` (visited on 07/04/2025).

[3] Jiří Velebil and Jiří Adámek. "A remark on conservative cocompletions of categories". In: *Journal of Pure and Applied Algebra* 168.1 (2002), pp. 107–124. ISSN: 0022-4049. DOI: `https://doi.org/10.1016/S0022-4049(01)00051-2`. URL: `https://www.sciencedirect.com/science/article/pii/S0022404901000512`.